Physics 308: Advanced Classical Mechanics
Fall 2007

Problem Set 10

Due: Fri 30 Nov 2007, in recitation section.

Reading: Please finish reading Chapter 9 in Taylor for Tuesday and read the first half of Chapter 11 for Thursday. The nonlinear dynamics reading for this week is Sec. 12.7.

Problems (continued on other side):

1. Two short problems on the Coriolis and centrifugal forces.
   (a) Taylor Problem 9.8. In this problem, you will determine the directions of the Coriolis and centrifugal forces for three different choices of velocity and location on the earth’s surface.
   (b) Taylor Problem 9.9. Here, you will compare the Coriolis force on a particularly fast moving projectile to the force of gravity. For slower objects, the Coriolis force will be proportionally smaller.

2. Parabolic surface of a spinning liquid. Taylor Problem 9.14. Please take as your definition of “equilibrium” the steady state solution in which each infinitesimal volume element $dV$ of water is at rest in the rotating frame of the bucket. Beyond what is asked in the textbook, please answer the following questions: (i) In the rotating frame, what are the forces on an infinitesimal volume element $dV$ of water at the surface? (You can neglect the surface tension of the water.) (ii) With respect to which forces is the surface an equipotential and why? (iii) In our discussion of central forces and the Kepler problem, we set $\ell = \text{const}$, and then defined an effective potential such that $F = -dU_{\text{eff}}(r)/dr$. In this problem, you want to do something similar, but setting $\omega = \text{const}$ instead. Why?

3. Puck on a merry-go-round: qualitative description. Taylor Problem 9.19. In this problem, you will take the dual points of view of an inertial and rotating frame, and describe the motion of an object initially at rest relative to one of those frames.

4. Puck on a merry-go-round: quantitative description. Taylor Problem 9.20. In this problem, you will work in the rotating frame of the merry-go-round and determine the precise trajectory of a puck released with arbitrary initial radius and velocity. (e) In addition
to what the textbook asks, please confirm that the resulting trajectory is a straight line when transformed back to an inertial frame.

5. **Accelerating truck with swinging door.** A truck moves with constant acceleration $a$ starting from rest at $t = 0$. On the rear of the truck a door of mass $m$ and length $\ell$ is free to swing as indicated in Fig. 1. The door is of uniform density and negligible thickness. The initial angle of the door is $\theta = 0$.

![Figure 1: Accelerating truck with swinging door.](image)

(a) Working in a noninertial frame in which the truck is at rest, draw a free body diagram indicating all forces, including “intertial forces” on the door.

(b) Please give the angular form of Newton’s second law for rotation of the door about the hinge $O$ in Fig. 1. (The moment of inertia of the door about $O$ is $I = m\ell^2/3$.) Then, solve for $\ddot{\theta}$. Answer:

$$\ddot{\theta} = \frac{3a}{2\ell} \cos \theta.$$  \hspace{1cm} (1)

(c) Since $\frac{d}{dt}(\dot{\theta}^2) = 2\ddot{\theta}$, we can multiply each side of Eq. (1) by $\dot{\theta}$ and then integrate to find $\dot{\theta}^2$ in terms of $\theta$. (You will have to determine the integration constant from initial conditions). Do this, and then solve for $\dot{\theta}$.

(d) At what time $t$ does the door swing shut? You can express your answer in terms of an integral, which you need not evaluate.

**Nonlinear dynamics problem:**

6. **State space orbits.** (a) Consider a simple pendulum consisting of a mass $m$ on the end of a rod of length $\ell$. The equation of motion is

$$\ddot{\phi} + \frac{g}{\ell} \sin \phi = 0,$$  \hspace{1cm} (2)

and the energy is

$$E = T + U = \frac{1}{2} m\ell^2 \dot{\phi}^2 + mg\ell(1 - \cos \phi).$$  \hspace{1cm} (3)
Please sketch the state-space orbits of the pendulum for the range $-2\pi \leq \phi \leq 2\pi$. Be sure to include orbits with $E < E_0$, $E = E_0$ and $E > E_0$, where $E_0 = 2mgl$. How would the state space orbits differ if we added a damping force?

(b) (Extra credit.) State space orbits are a broadly applicable tool for gaining insight into nonlinear differential equations that cannot be solved analytically. Here’s an example from cosmology:

In inflationary cosmology, the rate of expansion of the universe is given by the Hubble parameter

$$H = \sqrt{\frac{8\pi}{3} \left( \frac{1}{2} \dot{\varphi}^2 + U(\varphi) \right)^{1/2}}, \quad (4)$$

where $\varphi$ is the inflaton field, with equation of motion

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dU}{d\varphi} = 0. \quad (5)$$

If the inflaton is a free particle of mass $m$, then $U = \frac{1}{2}m^2\varphi^2$, and the last equation becomes

$$\ddot{\varphi} + \sqrt{\frac{12\pi}{3}} \left( \dot{\varphi}^2 + m^2\varphi^2 \right)^{1/2} \dot{\varphi} + m^2 \varphi = 0. \quad (6)$$

Please sketch the state-space $(\varphi, \dot{\varphi})$ orbits for the inflaton in this case. Hint: Since $\ddot{\varphi} = \dot{\varphi} \frac{d\dot{\varphi}}{d\varphi}$, we can write

$$\frac{d\dot{\varphi}}{d\varphi} = -\frac{\sqrt{12\pi} \left( \dot{\varphi}^2 + m^2\varphi^2 \right)^{1/2} \dot{\varphi} + m^2 \varphi}{\dot{\varphi}}. \quad (7)$$

This gives the slope of the orbit at any point $(\varphi, \dot{\varphi})$ in state-space.