Intersection Forms

PACT Feb 8.

Let X be a compact 4-manifold.

$$Q_X : H_2(X) \times H_2(X) \longrightarrow \mathbb{Z}.$$

$$Q_X([\Sigma_1], [\Sigma_2]) = |\Sigma_1 \cap \Sigma_2|.$$

↑ make transverse.

Ex

- $S^2 \times S^2$

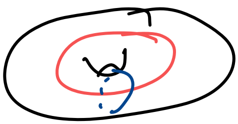
$$H_2 \cong \underset{\substack{\| \\ \langle [S^2 \times *] \rangle \\ A}}{\mathbb{Z}} \oplus \underset{\substack{\| \\ \langle [* \times S^2] \rangle \\ B}}{\mathbb{Z}}$$

$|A \cap A| = 0$

$|A \cap B| = 1$

$|B \cap B| = 0$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

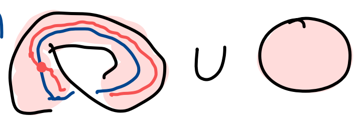


- $\mathbb{CP}^2$

$H_2 \cong \mathbb{Z} = \langle [\mathbb{CP}^1] \rangle$

$|[\mathbb{CP}^1] \cap [\mathbb{CP}^1]| = +1$

$[1]$



M^{2n}

$H_n(M) \times H_n(M)$

Properties

(1) Symmetric, bilinear.

(2) Intersection form vanishes on torsion elements

$$\langle\ ,\ \rangle : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}^{b_2(X)}}}{H_2(X)/TH_2(X)} \times \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}^{b_2(X)}}}{H_2(X)/TH_2(X)} \longrightarrow \mathbb{Z},$$

(3). If you pick a "basis" for $H_2(X)/TH_2(X)$,

$$\mathbb{Z}^{b_2 X} \times \mathbb{Z}^{b_2(X)} \to \mathbb{Z} \rightsquigarrow b_2(X) \times b_2(X) \text{ integer matrix}$$

$$\left|\det([Q_X])\right| = \mathrm{ord}(H_1(\partial X))$$

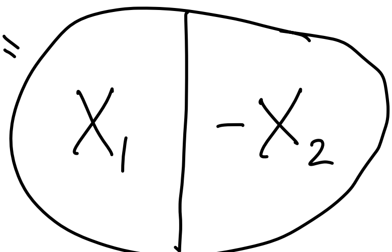
trivial group: 1

infinite groups: 0

(4) if X_1, X_2 are 4-mflds with

$\partial X_1 = Y = \partial X_2$ a $\underline{\mathbb{Q}HS^3}$, i.e. $H_1(X)$ finite

$$H_2(X;\mathbb{Q}) \cong H_2(X_1;\mathbb{Q}) \oplus H_2(X_2;\mathbb{Q})$$

$X =$ 

$$H_2(X_1)/TH_2(X_1) \oplus H_2(X_2)/TH_2(X_2) \hookrightarrow H_2(X)/TH_2(X)$$

---

connected sum of 4-mflds $\rightsquigarrow$ $\oplus$ of $\cap$ forms.

(5) Theorem [Donaldson] $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Z^4 closed, smooth 4-manifold.

If Q_Z is negative definite

$$Q_Z([\Sigma_i],[\Sigma_i]) < 0$$

for all $[\Sigma_i] \neq 0$.

then there is some "basis" for $H_2(Z)$ s.t.

$$[Q_Z] = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix}$$

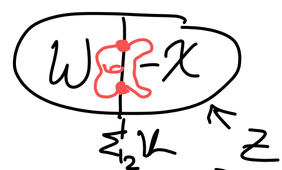
Cor No closed smooth 4-mfld has E_8 as its

Strategy WTS some knot K isn't slice.

Step 1: Find W s.t. $\Sigma_2^t K = \partial W^4$, Q_W neg. def not diag.

Step 2 Suppose K slice. $\Rightarrow \Sigma_2^t K = \partial X^4$, $H_2 X / TH_2 X = 0$.

Thm [Kjuchokova-M-Ray-Sakallı]



If $\sigma(K) = 0$, $u_{\mathbb{CP}^2}(K) \leq m$ [K is H-slice in $\#^m \mathbb{CP}^2 \times$]

$\mathbb{Z}^{10} \hookrightarrow \mathbb{Z}^{10}$

Then $\Sigma_2^t(K) = \partial X$, where $b_2(X) = 2m$,

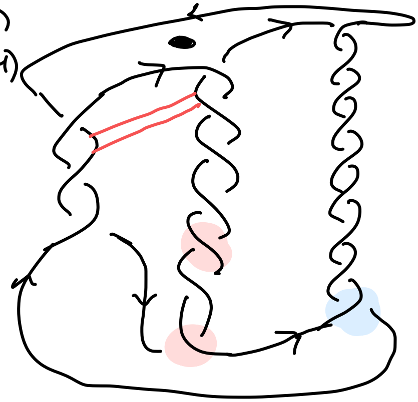
Q_X is pos def., can be rep. by a matrix

$$\begin{array}{c} \begin{bmatrix} 2I & I \\ I & * \end{bmatrix} \end{array} \begin{array}{l} m \\ m \end{array}.$$

K odd 3 str. pretzel:

K top slice $\Leftrightarrow$

- K sm. slice $K(p,-p,r)$, $K(1,q,-q-4)$
- $(\Delta_K(t)=1) \Rightarrow \det(K) \simeq 1$



Idea

Let $K = P(3,-7,9)$.

WTS: $u_{\mathbb{CP}^2}(K) = 2$
$u_{\overline{\mathbb{CP}^2}}(K) = 1$.

Quick part

- $u_{\mathbb{CP}^2}(K) \leq 2$

2 $+ \to -$ crossing changes

$P(3,-7,9) \rightsquigarrow \underbrace{P(3,-3,9)}_{\text{slice}}$

- $u_{\overline{\mathbb{CP}^2}}(K) \leq 1$

1 $- \to +$ crossing change

$P(3,-7,9) \to P(3,-7,7)$

- $u_{\overline{\mathbb{CP}^2}}(K) > 0$.

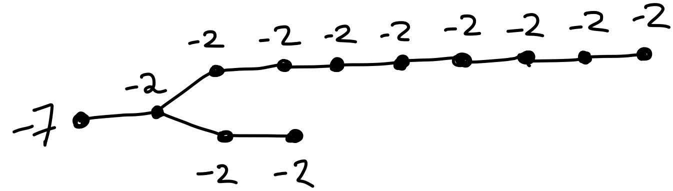
K is not slice $\Rightarrow$ $\det(K)$ not square.

Longer part: $u_{\mathbb{CP}^2}(K) > 1$.

$(S^4 \#^n \mathbb{CP}^2)^X$

Useful Fact

$\Sigma_2(K) = \partial$ (



) W

$K = P(3,-7,9)$