Slicing knots in definite 4-manifolds

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$\partial W = S^3$

Def'n

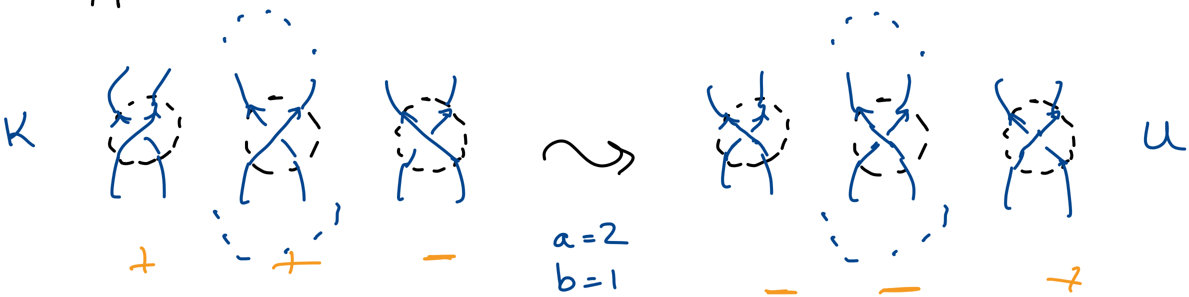
K a knot in S^3.

K is *slice* in a 4-mfld W if there is $D^2 \hookrightarrow B^4$ with $\partial D = K$.

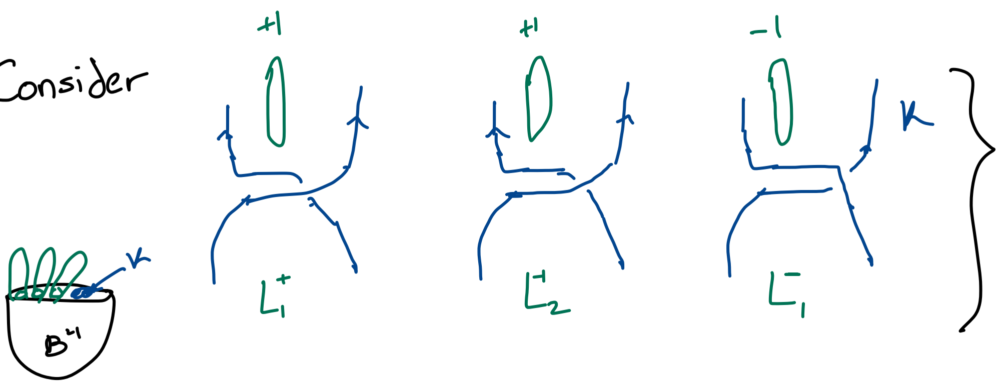
If X^4 is closed "slice in X" := "slice in $X \setminus \mathrm{int}\, B^4$"

Obs. 1 Every knot is slice in some W^4.

Suppose $K \rightsquigarrow U$ via $\begin{array}{ll} a & (+) \to (-) \\ b & (-) \to (+) \end{array}$ crossing changes.



Consider

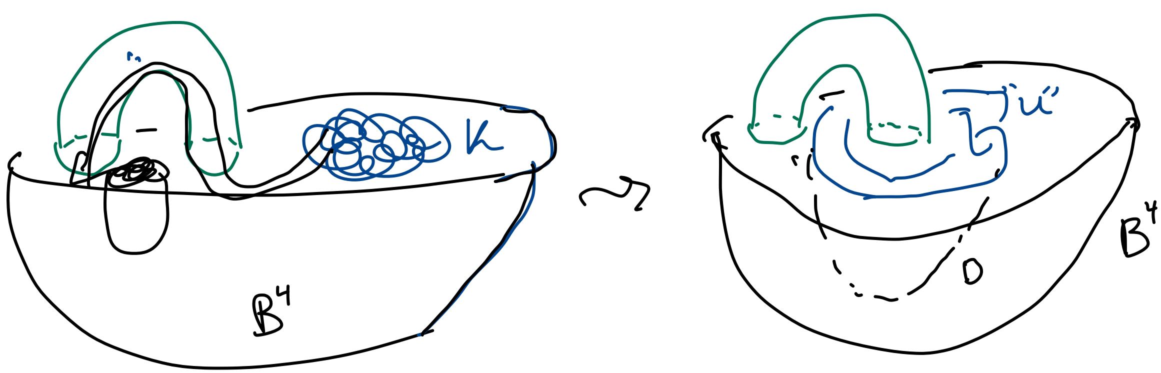


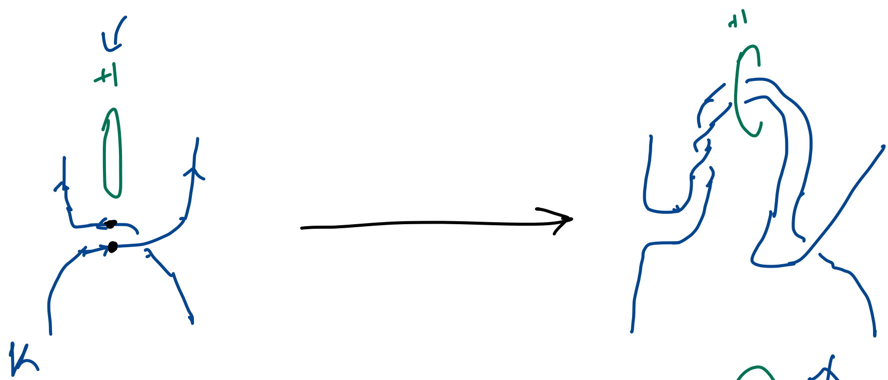
Observation 1: $\underline{W = \left(\#^a \mathbb{CP}^2 \# \#^b \overline{\mathbb{CP}^2}\right)^{\times}}$ $\leftarrow \partial W = S^3$

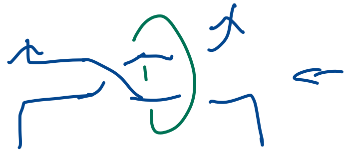
$$B^4 \cup (\mathbb{D}^2 \times \mathbb{D}^2) \cup \dots \cup (\mathbb{D}^2 \times \mathbb{D}^2)$$

$$\nu(L_i^{\pm}) \longleftrightarrow \partial \mathbb{D}^2 \times \mathbb{D}^2$$

Obs. 2 K is slice in W.







So K is H-slice in $(\#^a \mathbb{CP}^2 \# \#^b \overline{\mathbb{CP}^2})^\times$.

In fact.... the slice disc is null-homologous

So $T_{2,3}$ is H-slice in $\mathbb{CP}^2$.

~~$u_{\mathbb{CP}^2}(T_{2,3}) = 1$~~

[Exercise: $T_{2,3}$ is slice in $\overline{\mathbb{CP}^2}$] (but not H-slice.).

$u_{\overline{\mathbb{CP}^2}}(T_{2,3}) = \infty$

Def'n

- $u_{\mathbb{CP}^2}(K) = \min\{n \in \mathbb{N}_{\geq 0} : K \text{ is H-slice in } \#^n \mathbb{CP}^2\}$.
- $u_{\overline{\mathbb{CP}^2}}(K) = \min\{ \quad " \quad\quad " \quad \#^n \overline{\mathbb{CP}^2}\}$ (or ∞).

$$u_{\overline{\mathbb{CP}^2}}(K) = u_{\mathbb{CP}^2}(\overline{K}).$$

Thm [Cochran-Tweedy]

K is smooth H-slice in $\#^m \mathbb{CP}^2$ $\iff$ $K \sim J \longrightarrow R \leftarrow$ slice

$\sim$: smoothly conc.

$\longrightarrow$: m generalized $(+) \to (-)$ crossing changes

Obstructions, old and new

(1) K H-slice in $\#^m \mathbb{CP}^2 \Rightarrow -2m \leq \sigma_K(\omega) \leq 0$

$\#^n \overline{\mathbb{CP}^2} \Rightarrow 0 \leq \sigma_K(\omega) \leq 2n$

$\omega \in S^1$

Ex

$$u_{\mathbb{CP}^2}(\#^a T_{2,3}) = a \quad , \quad u_{\overline{\mathbb{CP}^2}}(\#^a T_{2,3}) = \infty$$

Two features

- $u_{\mathbb{CP}^2}(K),\ u_{\overline{\mathbb{CP}^2}}(K) < \infty \Rightarrow \sigma_K(\omega) \equiv 0$

  no bounds for $u_{\mathbb{CP}^2}$, $u_{\overline{\mathbb{CP}}^2}$
- this bound holds topologically

(2) $s(K), \tau(K), d(\Sigma_2(K))$, etc ... ?

their sign can imply $u_{\mathbb{CP}^2}(K) = \infty$

$u_{\mathbb{CP}^2}(T_{2,3}) = \underline{1} \Rightarrow u_{\mathbb{CP}^2}(C_{n,1}T_{2,3}) = \underline{1}$.

$C_{n,1}(U) = U$.

(3) Thm [Kjuchokova – M – Ray – Sakallı]

If K is smoothly H-slice in $\#^n \mathbb{CP}^2$, $\sigma(K) = 0$

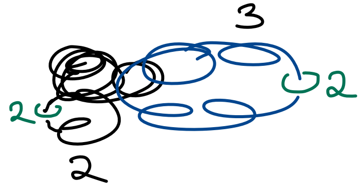
then $\Sigma_2(K) = \partial X^4$, where

(1) $b_2(X) = 2n$

(2) the intersection form of X is

(a) positive definite

(b) of half-integer surgery type

[i.e.
$$\begin{array}{c|cc} & n & n \\ \hline n & \boxed{2I} & I \\ n & I & \star \end{array} \longleftarrow$$


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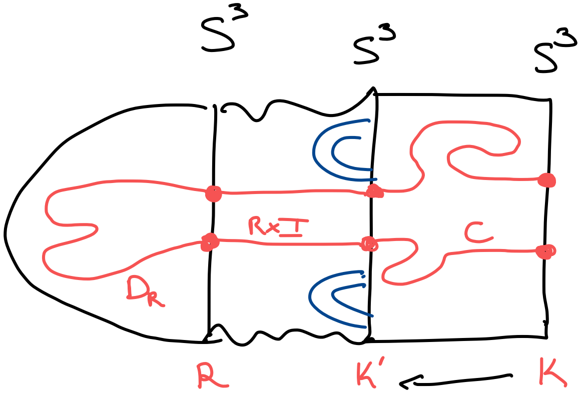
Idea of proof

$X = \Sigma_2\left(\#^n \mathbb{CP}^{2*}, \text{slice disc}\right).$

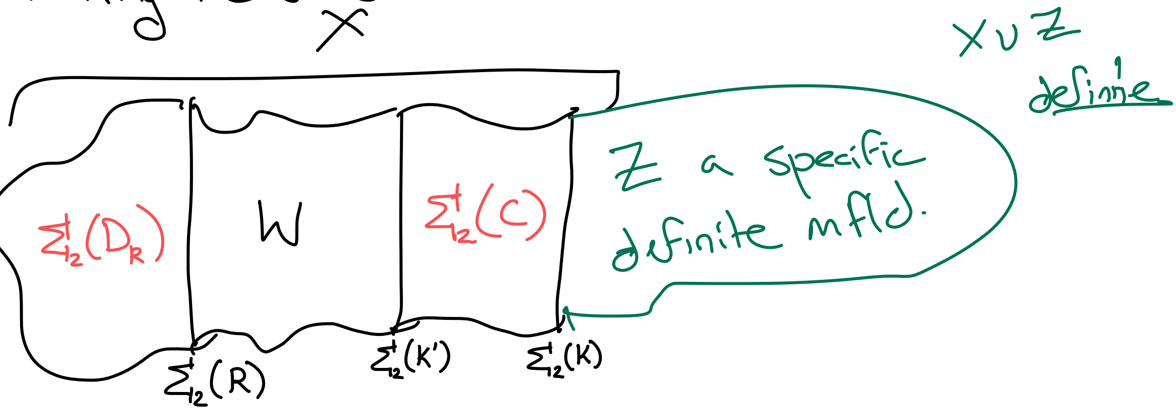
(1)+(2a): Already known

(2b):



$D \subseteq \left(\#^n \mathbb{CP}^2\right)^*$

Taking the double branched cover



$H_2(X) \cong H_2(W)$, $H_2(W)$ $\frac{1}{2}$. int. surgery type.

Why is this useful?

Idea:

$\varphi: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$

Q: Is there a "basis" $x_1, \dots, x_n$ for $\mathbb{Z}^n$ where this map looks like

$$\varphi(x_i, x_j) = \begin{cases} +1 & i = j \\ 0 & \text{else} \end{cases}.$$