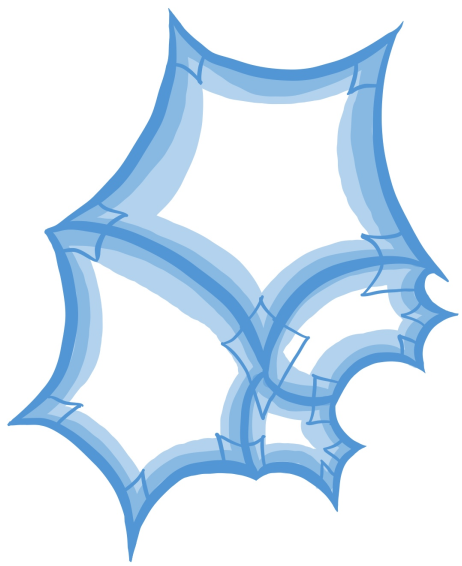
PATCH 2021

TALKS BY EMMY MURPHY AND EMILY STARK
NOTES LIVE-TEX'D BY THOMAS BRAZELTON AND LIVE-ILLUSTRATED BY MAXINE CALLE

Abstract. Notes from the Philadelphia Area interested in Topology, Contact/Symplectic Topology and Hyperbolic Geometry (PATCH) conference on October 1st, 2021 at Bryn Mawr College. Any errors or typos should be attributed to the note-takers and not the speakers.



Contents

1. Emily Stark: The Geometry of finitely generated groups

Abstract. In the 1980s Gromov proposed studying finitely generated groups as metric spaces. This perspective is powerful as groups that have similar large-scale geometry often share common algebraic features. In this introductory talk, we will present examples of this phenomena as well as tools to study the geometry of a finitely generated group.

Throughout this talk, we will assume that G is a finitely generated group. Suppose that G has a finite generating set S.

Definition 1.1. The *Cayley graph* for the group G, denoted $\mathrm{Cay}(G, S)$ which has vertex set $\{g \colon g \in G\}$, and two vertices are contained in an edge if they differ by a generator on the base $\{\{g, gs\} : g \in G,\ s \in S\}$. By the way the edges are defined, the group acts on the graph.

The Cayley graph $\mathrm{Cay}(G, S)$ is quasi-isometric to $\mathrm{Cay}(G, S')$ as long as S and S' are finite sets. The point is that the large-scale geometry of the group doesn't depend on a choice of generating set.

Example 1.2. Let $G = \langle S_1, S_2, S_3, S_4, S_5 \colon S_i^2 = 1,\ [S_i, S_{i+1}] = 1,\ i \pmod 5 \rangle$. We can draw the Cayley graph as:

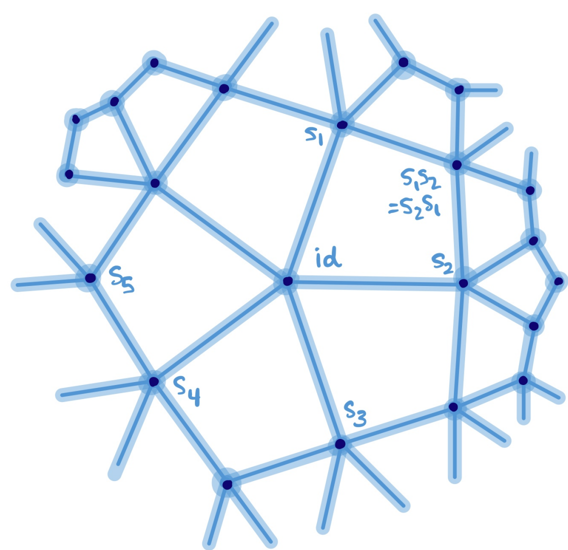


Figure 1. Caption

We have vertices for each generator, as well as vertices for products of generators, and so on. Every vertex will have valence 5 because there are five generators, and every vertex will be contained in four-cycles due to the relations.

The group G acts on the hyperbolic plane $\mathbb{H}^2$ by isometries, where every element of the group corresponds to reflection about a geodesic line in the hyperbolic plane.

The dual graph here is precisely the Cayley graph.

So we say that $\mathrm{Cay}(G, S)$ is quasi-isometric to $\mathbb{H}^2$ because $\mathbb{H}^2$ is a *model geometry* for G. By definition a *model geometry* for a group G is a proper[1], geodesic[2] metric space on which G acts geometrically. Here a geometric action means properly[3] and cocompactly[4] by isometries.

Definition 1.3. We say metric spaces X and Y are *quasi-isometric* if there exists a map $f : X \to Y$ and constants $k \geq 1$ and $C \geq 0$ so that

(1) For all $x, x' \in X$, we have that

$$\frac{1}{k} d(x, x') - C \leq d(f(x), f(x')) \leq k \cdot d(x, x') + C$$

This map distorts distances most by multiplicative and additive factors.

(2) (The map is almost onto): Everything in Y is within C of the image of f:

$$\mathrm{Neigh}_C(f(X)) = Y.$$

Example 1.4. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry.

Example 1.5. Any finite group is quasi-isometric to the trivial group, since its Cayley graph is compact.

Quasi-isometry invariants	**Algebraic consequences**
• The number of ends of X, defined to be the limit as $R \to \infty$ of the number of components in $X - B_R(p)$.	• A group G has two ends if and only if G contains $\mathbb{Z}$ as a finite index.
• δ-hyperbolicity (Gromov): Let $\delta \geq 0$. A metric space X is δ-*hyperbolic* if, for every geodesic triangle in X, the union of the δ-neighborhoods of any two sides contains the third.	• If G is δ-hyperbolic and not zero or two-ended, then G contains a free group F_n of rank $n \geq 2$ as a subgroup.

[1] A metric space X is *proper* if closed balls are compact

[2] A metric space X is *geodesic* if any two points are connected by a geodesic segment. A non-example would be $\mathbb{R}^2 - 0$. Here *geodesic* is induced by the path metric — meaning any path achieving the infimum of the length of paths between two points.

[3] A group action of G on a metric space X is *proper* if for all points $p \in X$ and constants $r \geq 0$, the cardinality $|\{g \in G \colon g \cdot B(r, p) \cap B(r, p) \neq \varnothing\}| < \infty$.

[4] A group action of G on a metric space X is *cocompact* if there is a compact subset $K \subseteq X$ so that the G-translates of K equal the entire space: $G \cdot K = X$.

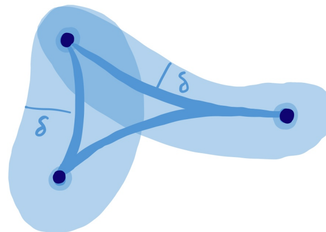


FIGURE 2.

We have zero-hyperbolicity exactly if and only if X is a tree.

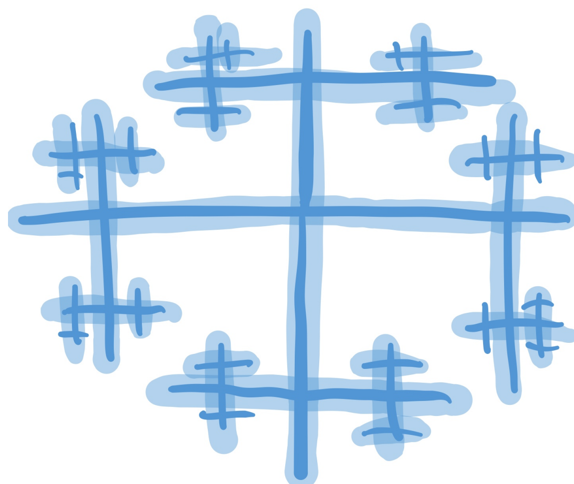


FIGURE 3.

Within the family of δ-hyperbolic groups, the *visual boundary* of G is a quasi-isometry invariant. Precisely, the visual boundary is defined by

$$\delta_\infty X := \{\text{equivalence classes of geodesic rays in } X\},$$

where two geodesic rays $\gamma, \gamma' : [0, \infty) \to X$ are equivalent if $d(\gamma(t), \gamma'(t)) \leq D$ for some $D \geq 0$ for all t. This comes equipped with a natural topology, where given any $z \in \delta_\infty X$, nearby geodesics to γ are representative geodesics passing through an open ball around z.

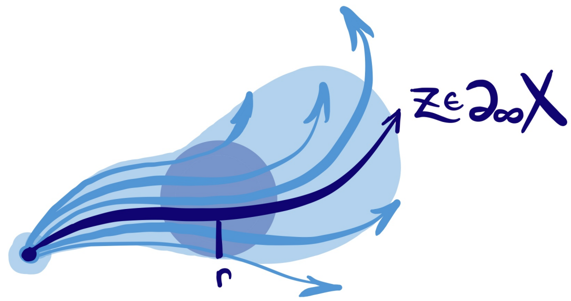


Figure 4.

For example:

- $\delta_\infty(\mathbb{H}^n) \cong S^{n-1}$.
- The visual boundary of a tree T_n for $n \geq 3$ is homeomorphic to the Cantor set.
- The visual boundary of a Bourdon Fuschian building[5]

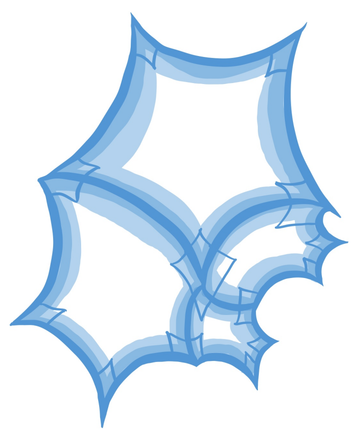


Figure 5.

is homeomorphic to the Menger curve.[6]

[5]These are 2-complexes where every cell is a regular right-angled hyperbolic p-gon, where all the sides are the same length.

[6]Recall the Menger curve is the fractal obtained by starting with $[0,1]^3$, then from every opposite faces we remove a square prism, then iterate this on each remaining cube. Cross-sections of this curve give the square Sierpinski carpet.

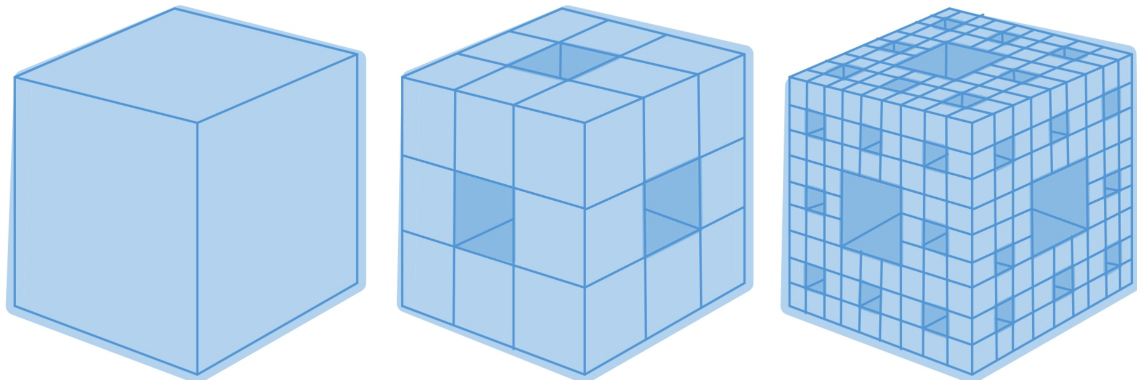


FIGURE 6.

Theorem 1.6. (Tukia, Gabai, Casson–Jungreis) If G is δ-hyperbolic, and $\delta_\infty(\mathrm{Cay}(G)) \cong S^1$[7], then G contains the fundamental group of a surface of genus at least two, $\pi_1(\Sigma_g)$, as a finite index subgroup.

Remark 1.7. A group is quasi-isometric to any of its finite-index subgroups.

Conjecture 1.8. (Cannon's Conjecture) If $\delta_\infty(G) \cong S^2$, then some finite index subgroup $\Gamma \subseteq G$ acts on $\mathbb{H}^3$ geometrically has boundary the 2-sphere.

Definition 1.9. We say that groups G and G' are *virtually isomorphic* if there exist finite normal subgroups $K \trianglelefteq G$ and $K' \trianglelefteq G'$ so that G/K and G'/K' contain isomorphic finite-index subgroups. Groups are *commensurable* if G and G' contain isomorphic finite-index subgroups.

If G and G' are virtually isomorphic, then they are quasi-isometric. The *rigidity problem* asks about the converse — when are quasi-isometric groups virtually isomorphic?

This is false in general (closed hyperbolic 3-manifold groups).

2. Emmy Murphy: Constructions of Liouville domains & etc.

ABSTRACT. We'll discuss the basics of Liouville manifolds and Weinstein handles. This is a method by which new symplectic manifolds can be constructed from old, using isotropic/Legendrian submanifolds of contact manifolds. We'll also discuss some of the ways this interacts with contact flexibility, namely loose Legendrians and overtwisted contact structures. These are tools by which, using some semi-local hypotheses, the geometric structures in question can be completely understood in terms of smooth topology.

Definition 2.1. A *symplectic structure* on a manifold is a 2-form $\omega \in \Omega^2(M)$ such that

[7]The visual boundary of any Cayley graph of G is well-defined, since it is a quasi-isometry invariant.

- It is closed ($d\omega = 0$)
- The map $\omega : TM \to T^*M$ is an isomorphism at each point.

It looks like the imaginary part of a Hermitian form: given some Hermitian metric $(\cdot, \cdot) = \langle \cdot, \cdot \rangle + i\omega(\cdot, \cdot)$.

Example 2.2. There are two main classes of examples that we will talk about.

(1) Consider any $X \subseteq \mathbb{C}^N$, embedded holomorphically in complex space (e.g. take a regular vanishing locus of polynomials). Then $\omega = \sum_{i=1}^N dx_j \wedge dy_j$.

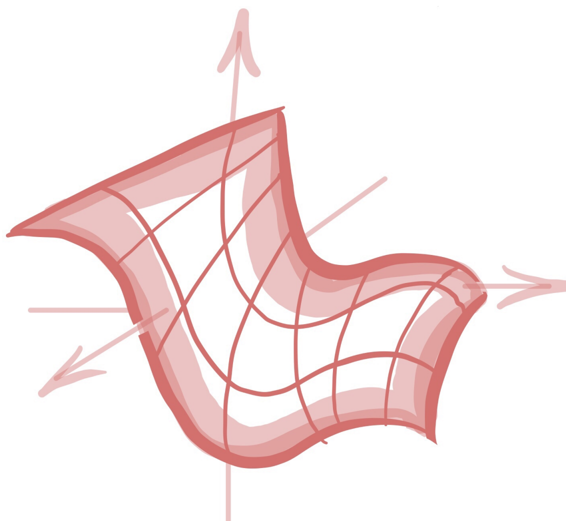


FIGURE 7.

(2) If Q is any smooth manifold, then T^*Q is symplectic, with 2-form given by
$$\omega = \sum dp_j \wedge dq_j,$$
where (q_j) are coordinates on Q and (p_j) are the dual coordinates in the fiber.[8]

This last example is particularly nice, in that it recovers all Hamiltonian mechanics on the manifold Q (i.e. this 2-form is what allows you to do Hamiltonian mechanics here).

What we want to talk about are specific types of symplectic manifolds called Liouville domains/manifolds. People often study compact closed symplectic manifolds, which we don't want to do.

Definition 2.3. A symplectic structure is *exact* if $\omega = d\lambda$ for some $\lambda \in \Omega^1(M)$.

[8]"$p_j = dq_j$" as a fiber coordinate on T^*Q. More explicitly, $dq_j \in \Omega^1 Q$, and using the projection $\pi : T^*Q \to Q$, we can get a pullback $\pi^*(dq_j) \in \Omega^1(T^*Q)$.

This gives you something for free — since $\omega : TM \to T^*M$ was an isomorphism, the data of λ defines Z_λ (a vector field) by the property that $\omega(Z_\lambda, \cdot) = \lambda$.

Definition 2.4. A *Liouville domain* is an exact symplectic manifold, which is compact with boundary, so that Z_λ is transverse to the boundary ∂M.[9]

FIGURE 8.

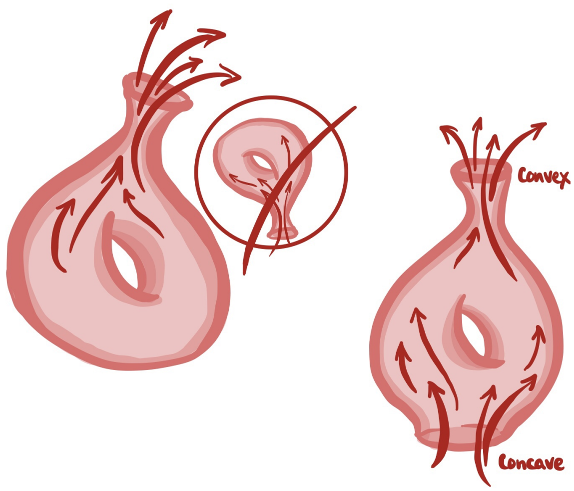


FIGURE 9.

[9]If ∂M is connected, where Z_λ meets the boundary needs to be "outwardly transverse." If $\phi_t : M \to M$ is the flow of Z_λ, then following from the equation $\omega(Z_\lambda, \cdot) = \lambda$, we get that $\phi_t^* \omega = e^t \omega$ is expanding the structure. Also if we take the top-dimensional wedge power of ω, being $\omega^n \in \Omega^{2n}(M)$, we have that this is a volume form. Thus the flow is exponentially expanding the volume, so we couldn't have a manifold where we are flowing into it and expanding the volume.

The outward transversality is called *convex Liouville*, while inward transversality is called *concave Liouville*.

Remark 2.5. Both of these examples from before fall under this definition.

Example 2.6. If $M = T^*Q$, we can take $\lambda = \sum p_j \, dq_j$, and $Z_\lambda = \sum p_j \partial_{p_j}$.[10] If $\widehat{M} = \{||p|| \leq 1\} \subseteq T^*Q$, then $\widehat{M}$ is Liouville.

Example 2.7. If $X \subseteq \mathbb{C}^N$ is holomorphically and properly embedded[11] then $X \cap B^{2N}(R)$ is Liouville, where R is some large radius.

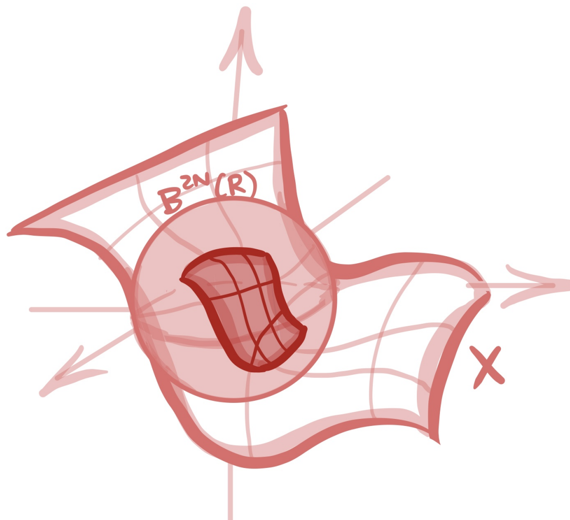


Figure 10.

This is because we could take $\lambda = \frac{1}{2} \sum x_j \, dy_j - y_j \, dx_j$, and $Z_\lambda = \frac{1}{2} \sum x_j \partial_{x_j} + y_j \partial_{y_j}$.[12]

This definition is broad enough so that it incorporates a lot of holomorphic geometry, as well as a lot of Hamiltonian mechanics.

Question: Why do we require this "transverse to the boundary" condition?

Pseudo–answer: We need *something* at the boundary to make the geometry interesting. We would have too much freedom without this, and we would have no control over how the geometry is acting near the boundary.

Aside: If we take $\mathbb{C}^2 - 0$, (or $\mathbb{C}^k - 0$ for $k \geq 2$) then there does not exist a biholomorphism $f : \mathbb{C}^2 - 0 \to \mathbb{C}^2 - 0$ interchanging 0 and ∞. This isn't true for $\mathbb{C} - 0$ obviously — roughly due to the fact that S^1 is flat, while higher spheres are not.

[10] This Z_λ is the fiberwise radial vector field in each fiber.

[11] Think of affine varieties.

[12] Really we have to take the orthogonal projection onto X of this vector field in order to have transversality at the boundary.

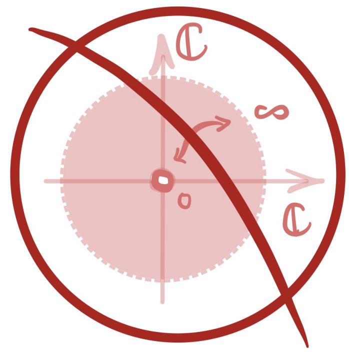


FIGURE 11.

We want to talk about how to construct Liouville domains. Given a smooth manifold with boundary ∂M, does there exist a Liouville structure on it? Give me tools to build stuff.

Weird example: There exists M^4 Liouville with topology given by the unit sphere bundle cross an interval $S\left(T^*\Sigma_{g\geq 2}\right)\times[0,1]$, union a 1-handle if you want connected boundary. This is not a *Stein* manifold, since the sphere bundle is a 3-manifold and has nontrivial H^3.

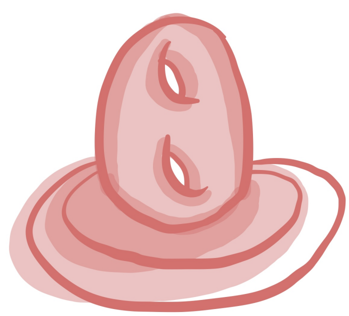


FIGURE 12.

Handle attachments In smooth topology, given some manifold M (compact with boundary), and given some $(k-1)$-sphere $S^{k-1}\subseteq\partial M$ with trivial normal bundle, then we can define a new manifold $\widetilde{M}:=M\cup_{S^{k-1}\times D^{n-k}}\left(D^k\times D^{n-k}\right)$, where $S^{k-1}\subseteq D^{n-k}\subseteq\partial M$ and also a subset of $\partial\left(D^n\times D^{n-k}\right)$. For example if we wanted to build a torus, we could take a bowl and attach two handles onto it — we see that this is homeomorphic to a torus with a disk chopped out (exercise: visualize this). If we union on a copy of $D^2\times D^0$, we get T^2.

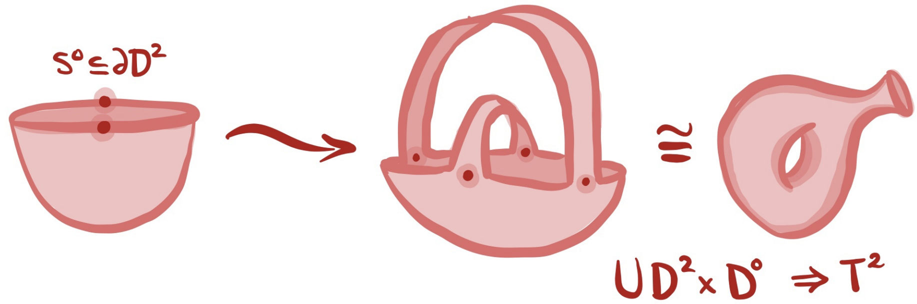


FIGURE 13.

A classical result is that all smooth manifolds can be built this way.

Question: Is there an analogue for Liouville domains?

Answer: Yes! This is what is called *Weinstein handle attachment*.

The subtlety comes from gluing along a submanifold in the boundary. We need to understand what conditions on the submanifold in ∂M are we going to need in order to glue the geometry consistently. This gets into something called *contact geometry*.

Following from the property that Z_λ is transverse to ∂M, this implies that

$$\lambda \wedge \underbrace{d\lambda \wedge d\lambda \wedge \cdots \wedge d\lambda}_{n-1}$$

is a volume form in $\Omega^{2n-1}(\partial M)$. The form λ is locally given by $\lambda = dz - \sum_{j=1}^{n-1} y_j dx_j$. It turns out then that as long as the submanifold $\Lambda = S^{k-1} \subseteq \partial M$ satisfies that $\lambda|_\Lambda = 0$, then there is a well-defined Liouville handle attachment.

First, we see that any S^0 works, so we can attach 1-handles. In general, $\lambda|_{S^{k-1}} = 0$ implies that $k - 1 \leq n - 1 = \dim \partial M$. Thus we can only attach handles with $\leq \frac{1}{2}$-dimensional homotopy type.[13]

Drawing in the xz-plane, we can always solve the equation $\lambda = dz - \sum_{j=1}^{n-1} y_j dx_j$ by letting $y = \frac{\partial z}{\partial x}$. Let $S^1 \subseteq \mathbb{R}^3 \subseteq S^3$ be the following:

[13]This vanishing happens if and only if $S^{k-1} \times (-\varepsilon, \varepsilon)$ (which is thickened using the flow) is *Lagrangian/isotropic*, meaning that $\omega|_{S^{k-1} \times (-\varepsilon,\varepsilon)} = 0$.

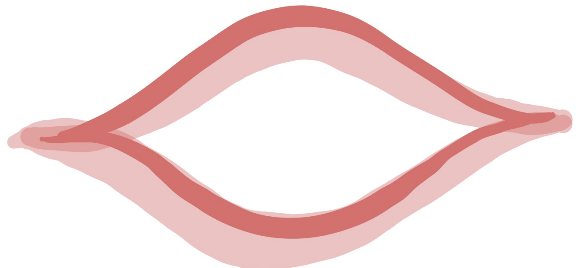


FIGURE 14.

If we take B^4 union a 2-handle, this gives the manifold $D(T^*S^2)$, which is also equal to the affine quadric $\{x^2 + y^2 + z^2 = 1 \quad \cap B^6(R) \subseteq \mathbb{C}^3$.

As another example, consider the following:

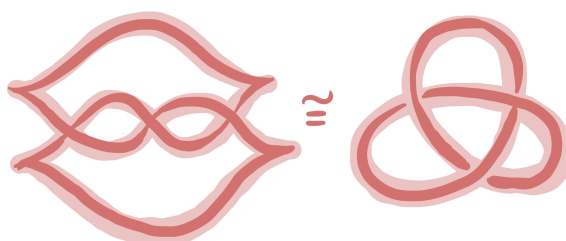


FIGURE 15.

Taking B^4 union a 2-handle, we are getting $\{xyz + x + z + 1 = 0\}$.

Finally, consider two spheres with 1-handles drawn:

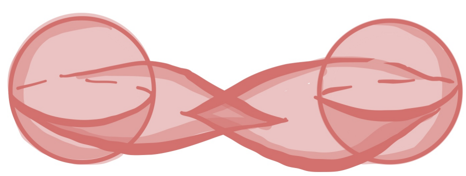


FIGURE 16.

Again after gluing, this is equal to $\mathbb{C}^2 - \{xy = 1\}$.

3. Emily Stark: Graphical discrete groups & rigidity

Abstract. Rigidity theorems prove that a group's geometry determines its algebra, typically up to virtual isomorphism. Motivated by interest in rigidity, we study the family of graphically discrete groups. In this talk, we will present rigidity consequences for groups in this family. We will present classic examples as well as new results that imply this property is not a quasi-isometry invariant. This is joint work with Alex Margolis, Sam Shepherd, and Daniel Woodhouse.

We will give two equivalent definitions: from lattice embeddings, and from groups acting on graphs.

Definition 3.1. A finitely generated group Γ is *graphically discrete* if whenever Γ/K is isomorphic to a uniform lattice in a totally disconnected locally compact second countable toplogical group G, where $K \trianglelefteq \Gamma$ is finite, then G is compact-by-discrete.[14]

Note 3.2. If Γ acts geometrically on X, where X is a locally finite graph, then Γ modulo the kernel of the action embeds into the isometry group $\mathrm{Isom}(X)$ as a uniform lattice, and $\mathrm{Isom}(X)$ is a totally disconnected locally compact group. This gives a concrete example of the first half of the previous definition.

Then $\mathrm{Isom}(X)$ is discrete if no nontrivial sequence converges to the identity. For example the isometry group of the integer lattice $\mathbb{Z}^2$ is discrete, but the isometry group of a tree is not.

Definition 3.3. Let X be a graph. An $\mathrm{Aut}(X)$-*imprimitivity system* on X is an equivalence relation on the vertex set of X that is invariant under $\mathrm{Aut}(X)$. That is, $v \sim w$ if and only if $h \cdot v \sim h \cdot w$ for all $h \in \mathrm{Aut}(X)$. Equivalence classes are called *blocks*.

Example 3.4. Let X be the integer tiling of the line with two leaves attached to every vertex. Note that the isometry group $\mathrm{Isom}(X)$ is **not** discrete, since we can flip the leaves at each point and get a non-trivial sequence converging to the identity. We can build a quotient graph $X/\sim$, where the vertex set is in bijection with equivalence classes, and two equivalence classes $[v]$ and $[w]$ share an edge in the quotient graph if there are vertices $v' \in [v]$ and $w' \in [w]$ that are an edge in the original graph. The quotient $X/\sim$ here would just be a line with vertices at each integer, where $\mathrm{Aut}(X)$ acts on it discretely[15].

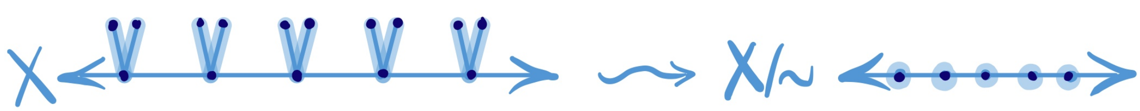


Figure 17.

[14] G surjects onto a discrete group, and the kernel is compact

[15] The image of $\mathrm{Aut}(X)$ in the automorphism group of the quotient graph is discrete.

Definition 3.5. A finitely generated group Γ is *graphically discrete* if, whenever Γ acts geometrically on a locally finite graph X, there exists an $\mathrm{Aut}(X)$-imprimitivity system with finite blocks, so that the induced action of $\mathrm{Aut}(X)$ on $X/\sim$ is discrete.

Remark 3.6. (On rigidity) If Γ_1 and Γ_2 act geometrically on a locally finite graph X, and Γ_1 is graphically discrete, then Γ_1 and Γ_2 are virtually isomorphic. Furthermore, their images in $\mathrm{Aut}(X/\sim)$ are commensurable.

Theorem 3.7. (MSSW) If Γ is a free product of residually finite[16] graphically discrete groups, then Γ is *action rigid*[17] within residually finite groups.

Rather than focus on the proof, we will discuss examples of graphically discrete groups.

Example 3.8. (Trofimov, 1984) Virtually nilpotent groups are graphically discrete. For example $\mathbb{Z}^n$, and the integer Heisenberg group.

Example 3.9. (Bader–Furman–Sauer, 2020) Irreducible lattices in connected center-free real semisimple Lie groups with no compact factors, except for nonuniform lattices in $\mathrm{PSL}_2(\mathbb{R})$.

3.1. **Boundary criterion.**

Proposition 3.10. Let G be a δ-hyperbolic group. Then G is graphically discrete if, whenever G acts geometrically on a locally finite graph X and for all $x \in X^{(0)}$, we have that

$$(\mathrm{Aut}(X))_{[x]} \to \mathrm{Homeo}(\partial X)$$

has finite image.

This can be used to show that groups are *not* graphically discrete, by exhibiting a suitable action and vertex.

Corollary 3.11. Free groups are *not* graphically discrete, as they act geometrically on a tree, but the stabilizer of the center vertex allows you to flip anything away from the center — in particular there are infinitely many such automorphisms.

Similarly lattices in Bourdon Fuschian buildings are not graphically discrete, because of the huge automorphism group of the building.

3.2. **3-manifold groups.**

Let M be a connected closed orientable irreducible 3-manifold. Then Perelman's proof of Thurston's geometrization conjecture implies that M is either "geometric"[18] or it is "non-geometric" and decomposes along a family of JSJ tori into hyperbolic and Seifert fibered components.

[16] The intersection of all finite-index subgroups is the identity.

[17] If Γ and Γ' act geometrically on the same proper geodesic metric space, then Γ and Γ' are virtually isomorphic.

[18] It admits one of Thurston's eight three-dimensional geometries.

Theorem 3.12. (Trofimov, Bader–Furman–Sauer, Dymarz; Kapovich–Leeb) If M is geometric, then its fundamental group $\pi_1(M)$ is graphically discrete.

Theorem 3.13. (MSSW) If M is non-geometric, then its fundamental group is graphically discrete.

Proof ideas:

- If M is non-geometric, the JSJ tori in M yield a graph of spaces decomposition of M with JSJ tree T.

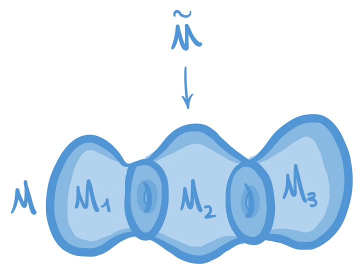


FIGURE 18.

- Suppose that $\pi_1(M)$ acts geometrically on some locally finite graph X. Then the quasi-isometry invariants of the JSJ decomposition due to Kapovich–Leeb proves that $\mathrm{Aut}(X)$ also acts on the Baser(sp?) tree T. We can use this to show that X has a "coarse tree of spaces" decomposition mimicking $\widetilde{M}$ and invariant under $\mathrm{Aut}(X)$.

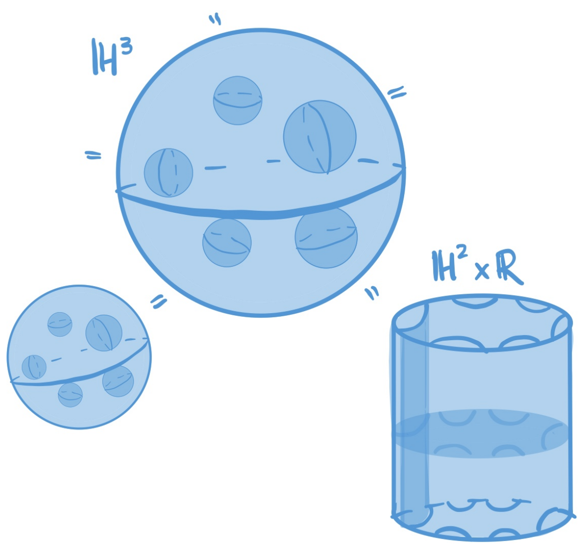


FIGURE 19.

- **Imprimitivity system**: Replace these "edge spaces" with R-neighborhoods so that the graph X is equal to the union of the new edge spaces $\cup_{e \in ET} X_e$. To write down an equivalence class, we say that $x \sim x'$ if the set of edge spaces containing x is equal to the set of edge spaces containing x'. Then, $\mathrm{Aut}(X)$ preserves this equivalence relation, and the equivalence classes are finite. This is the standard way to build an imprimitivity system from a graph of spaces.
- Roughly, the action is discrete because if an automorphism fixes an edge space, it fixes the entire space.

4. Emmy Murphy: Liouville cobordisms

Abstract. In this talk we'll discuss some interesting Liouville cobordisms arising in the particular case when the negative boundary is an overtwisted contact manifold. This will center on two independent constructions: concordances in the high-dimensional setting, and cobordisms with high-index (and therefore non-Weinstein) topological type.

Given a Liouville domain M^{2n} (compact, with boundary), there exists Weinstein handle attachments along *isotropic*[19] spheres $S^{k-1} \subseteq \partial M$. It is required that $k \leq n$ in order to be isotropic. We can only construct manifolds whose homotopy type is $\leq \frac{1}{2}\dim(M)$.

Question: To what extent does the converse of this statement hold? Meaning if M is a smooth manifold with $\frac{1}{2}$-dimensional homotopy type, can it be built with this Liouville construction?

Answer: In general the answer is no. The main counterexample is the manifold $S^2 \times D^2$, which doesn't admit a Liouville structure.

Contrast this with the following — in dimensions $2n \geq 6$, any smooth manifold with half homotopy type and a complex structure (on TM as a vector bundle) can be built out of Weinstein handles (Eliashberg, '90). We can think of this complex structure on TM as the existence of a lift

$$\begin{tikzcd} & \mathrm{BU}_n \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "TM"'] & \mathrm{BO}_{2n}. \end{tikzcd}$$

We want to try to generalize this to not just talk about domains themselves, but about cobordisms.

Recall we have this vector field Z_λ which is required to be transverse to the boundary, and it could be inward or outward flowing. Denote by $\partial_- M$ the set where it is inwardly transverse, and $\partial_+ M$ the set where it is outwardly transverse.

[19] Recall this means that $\lambda|_{S^{k-1}} = 0$

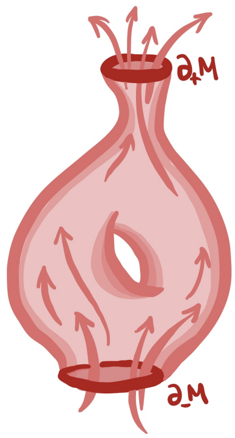


FIGURE 20.

Note 4.1. The positive boundary $\partial_+ M$ is always nonempty due to volume conditions.

To what extent does this depend on the *contact geometry* at $\partial_\pm M$? Whether or not it is possible to glue handles onto $\partial_- M$ may depend on not only the topology, but on the *geometry* at $\partial_- M$.

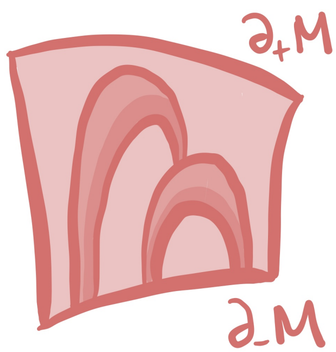


FIGURE 21.

Among contact manifolds, there are two types of contact structures:

- *Overtwisted manifolds*: in some ways these are the nicest contact structures you could ever work with, and in some ways they are the worst. We have a much better understanding of what possibilities may occur here. Diffeomorphism plus a framing conditions implies contactomorphism in the setting of overtwisted manifolds. Another important property of overtwisted structures is that any codimension zero smooth embedding of a contact manifold is isotopic to a contact embedding (again in the presence of a framing condition). So not only can we completely classify overtwisted manifolds, we can understand open sets — since we can just try to smoothly embed them in overtwisted manifolds.[20]

- *Tight contact manifolds*: this is everything else. In the case where $\partial_- M = \varnothing$, then $\partial_+ M$ is tight. In some sense these are the nicer ones, since this is what you run into in nature.

Given any cobordism M (we are viewing M as a cobordism between $\partial_+ M$ and $\partial_- M$) with TM having a complex structure so that $(M, \partial_- M)$ has $\frac{1}{2}$-dimensional homotopy type, then there exists a canonical Liouville structure on M such that $\partial_\pm M$ are both overtwisted. This is called a *flexible cobordism*.

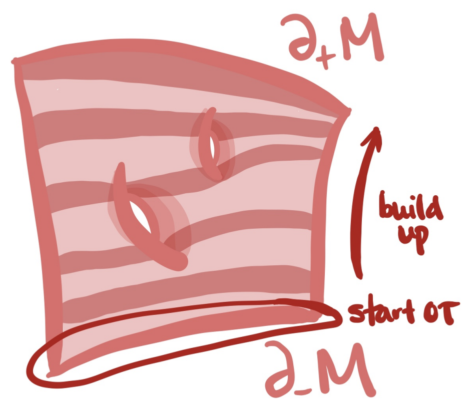


FIGURE 22.

For contrast, there does not exist any Liouville manifold M^3 such that $\partial_- M = S^3_{\text{tight}}$ and $\partial_+ M = S^3_{\text{OT}}$. This is due to Gromov in '85.

[20]The discussion of overtwisted manifolds is due to Eliashberg '89 for 3-dimensions, and Borman–Eliashberg–Murphy in '15 for ≥ 50 dimensions.

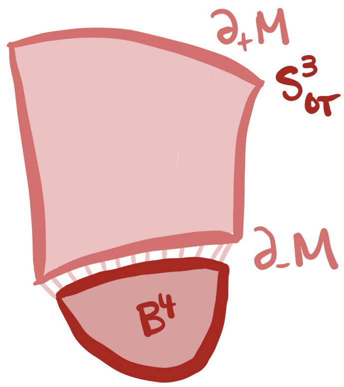


FIGURE 23.

There also does not exist a Liouville structure on $S^3 \times [0,1] = M$ such that $\partial_- M = S^3_{\mathrm{OT}}$ and $\partial_+ M = S^3_{\mathrm{tight}}$. However for other topologies of M, this is possible.

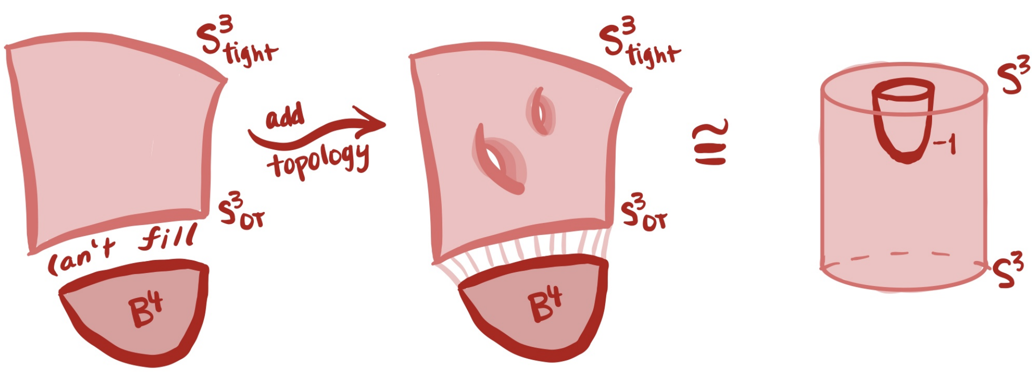


FIGURE 24.

In higher dimensions, given *any* contact structure ξ on Y, there exists a Liouville structure on $Y \times [0,1] = M$ such that $\partial_- M = (Y, \xi_{\mathrm{OT}})$ and $\partial_+ M = (Y, \xi_{\mathrm{tight}})$.

Thus there is some difference between the lower-dimensional and the higher dimensional cases, really depending upon the topology.

Question: What about higher index homotopy types? That is, if $H^*(M, \partial_- M) \neq 0$ for $* > n$?

Here things are completely unconstrained as well.

Theorem 4.2. (E-M): Let M be any smooth manifold with a complex structure on TM, and suppose that $\partial_+ M$ and $\partial_- M$ are non-empty (so we are talking about an actual cobordism). Then there is a Liouville structure on M so that $\partial_\pm M$ are both overtwisted.[21]

This doesn't depend on dimensions, so we can patch it together with the previous result (stick the concordance $Y \times [0,1]$ on top, and assume that $\partial_+ M$ is whatever you want it to be).

Rough idea: We want to build a high-index handle. In, for example $M = D^k \times D^{n-k}$, we can find a codimension two submanifold Σ so that $M - \Sigma$ has half-dimensional homotopy type

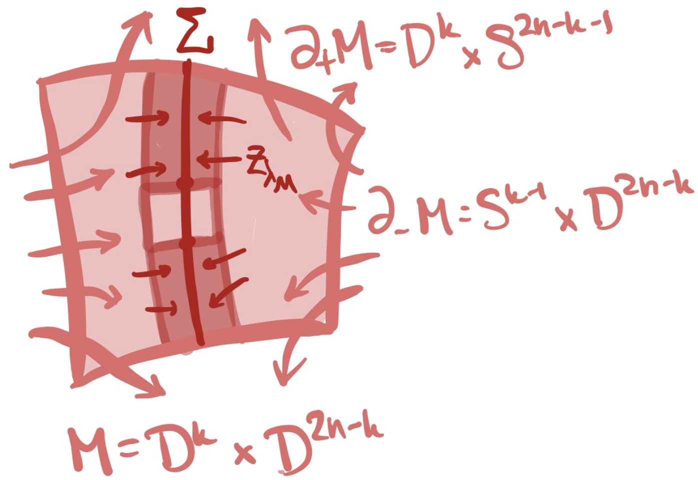


FIGURE 25.

Then $\Sigma - D^{2n-k(?)}$ has a Liouville structure. Near Σ, we have that $\lambda_M = (r^2-1)d\theta + \lambda_\Sigma$. This is not defined at $r=0$, but $d\lambda_M$ is. We will have that $Z_{\lambda_M} = \left(r - \frac{1}{r}\right) dr$. At the end, we add $d\theta$ back to λ_M. This gives a different Liouville structure, but it doesn't change $d\lambda_M$.

[21]Note that there is no half-dimensional homotopy type hypothesis in this result.